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FAST TRACK COMMUNICATION

Fractal Lévy correlation cascades

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Abstract

The correlation structure of a wide class of random processes, driven by stable non-Gaussian Lévy noise sources, is explored. Since these noises are of infinite variance, correlations cannot be measured via auto-covariance functions. Exploiting the underlying Poissonian structure of Lévy noises, we present a cascade of 'Poissonian correlation functions' which characterize the correlation structure and the process distribution of the processes under consideration. The theory developed is applied to various examples including motions, Ornstein–Uhlenbeck and moving-average processes, and fractional motions and noises—all driven by stable non-Gaussian Lévy noises.

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1. Introduction

A wide range of random processes can be represented as an integral transform of a 'driving' random noise. Specifically,

$$X(t) = \int_{-\infty}^{\infty} K(t; s) N(\mathrm{d}s) \tag{1}$$

where (i) $X = (X(t))_t$ is the random process under consideration; (ii) $N = (N(t))_t$ is the driving random noise and (iii) $K(t; s)$ is a non-negative-valued integration kernel, governing the integral transform.

Equation (1) can be interpreted as a system transforming an *input noise* into an *output process*. A key feature of this transformation is that it induces *correlation*: the integration with respect to the kernel $K(t; s)$ convolutes *uncorrelated* input noises into *correlated* output processes.

Examples of processes driven by white noise and admitting the integral representation of equation (1) include: Brownian motion [1]; Ornstein–Uhlenbeck motions [2] and their moving-average generalizations [3] and fractional Brownian motions and noises [4].

For long years white noise served as the dominant model-of-choice for randomness in stochastic systems. This choice was based on solid grounds: the central limit theorem (CLT) [5]. The CLT asserts that white noise is the only possible scaling limit of uncorrelated noises with finite variance. The CLT explains the widespread appearance of white noise and of its universal hallmark—the Gaussian distribution. However, the CLT fails to hold when variances are infinite. In such cases, the CLT is replaced by the generalized CLT—due to Gnedenko, Kolmogorov and Lévy—asserting that the only possible scaling limits are *stable Lévy noises* [6, 7].

The class of Lévy noises is characterized by the property of 'independent and stationary increments' [8–10]. The independence-of-increments is the manifestation of 'pure noise': gathering information about one noise increment discloses absolutely no information about other noise increments. The stationarity-of-increments is, in fact, a statistical *shift invariance*. Stable Lévy noises share the additional property of statistical *scale invariance* ('self-similarity', 'fractality'). White noise is the only stable Lévy noise possessing finite variances.

In recent years Lévy noises have drawn much attention and research. Numerous examples of Lévy-type statistics have been empirically observed in various areas including anomalous diffusion [11, 12], heartbeats [13], firing of neural networks [14], seismic activity [15], signal processing [16] and financial time series [17, 18]. The ruling paradigm of Gaussian noise modelling in stochastic systems began to give way to the examination and incorporation of models driven and perturbed by Lévy noise sources [19–24].

If we take the input N to be a symmetric stable Lévy noise—the natural generalization of white noise—then the Fourier transforms of the multidimensional marginal distributions of the output process X are given by [8]

$$\left\langle \exp\left\{ \mathrm{i}\left(\sum_{j=1}^{n} \omega_j X(t_j) \right) \right\} \right\rangle = \exp\left\{ -a \int_{-\infty}^{\infty} \left| \sum_{j=1}^{n} \omega_j K(t_j; s) \right|^{\alpha} \mathrm{d}s \right\}, \qquad (2)$$

where (i) $\{t_j\}_{j=1}^{n}$ are arbitrary time points and $\{\omega_j\}_{j=1}^{n}$ are the corresponding Fourier variables and (ii) the parameters α and a $(0 < \alpha \leqslant 2\,;\ a > 0)$ are, respectively, the *Lévy exponent* and the *amplitude* of the input noise.

The case of white noise with standard deviation σ $(\sigma > 0)$ corresponds to the Lévy exponent $\alpha = 2$ and to the amplitude $a = \sigma^2/2$. In this case, the output X is a *Gaussian process*, and equation (2) reduces to

$$\left\langle \exp\left\{ \mathrm{i}\left(\sum_{j=1}^{n} \omega_j X(t_j) \right) \right\} \right\rangle = \exp\left\{ -\frac{1}{2} \sum_{j_1, j_2=1}^{n} C(t_{j_1}, t_{j_2}) \omega_{j_1} \omega_{j_2} \right\}, \qquad (3)$$

where $C(t', t'') = \sigma^2 \int_{-\infty}^{\infty} K(t'; s) K(t''; s)\, \mathrm{d}s$ is the output's *auto-covariance* function. Equation (3) implies that the output's process distribution is *characterized* by its auto-covariance function—a special feature *unique* to Gaussian processes.

The non-Gaussian case—corresponding to Lévy exponents in the range $0 < \alpha < 2$—is dramatically different than the Gaussian case. In the non-Gaussian case variances are infinite, and hence the output processes posses no auto-covariance functions. Moreover, the multidimensional Fourier transforms of equation (2) are of little practical use. Thus, the correlation structure of non-Gaussian outputs is far less tractable and amenable to mathematical analysis than that of Gaussian outputs.

Two measures of correlation, devised to tackle the non-Gaussian case, are the *covariation* and the *codifference* [8]. The former is defined in the parameter sub-range $1 < \alpha \leqslant 2$, whereas the latter is defined in the entire parameter range $0 < \alpha \leqslant 2$. Both measures are based on the spectral structure of stable Lévy noises and reduce to the 'standard' covariance at the Gaussian endpoint $\alpha = 2$. However, these measures of correlation fail to characterize the output's process distribution: two different output processes may possess either the same covariation or the same codifference.

In what follows we present a correlation analysis of the output processes—applicable to the non-Gaussian case—which is based on the underlying *Poissonian structure*, rather than on the spectral structure, of the inputs and outputs. This Poisson-based analysis gives rise to the *fractal Lévy correlation cascade*—an infinite cascade of 'Poissonian correlation functions' which *characterize* the output's process distribution. This cascade turns out to be the 'Lévy counterpart' of the auto-covariance function—in the passage from white noise input to symmetric stable Lévy noise inputs.

The theory of Lévy correlation cascades, in the context of general (not necessarily stable) Lévy noises, is developed in [25]. An analogous theory for general shot noise processes is presented in [26]. Readers interested in the chaotic properties (ergodicity, mixing) of the processes under consideration are referred to [27, 28].

2. Poissonian analysis

Our analysis is based on a deep connection between the theory of Lévy distributions and processes, and the theory of Poisson point processes [29]—known, in the literature, as the celebrated Lévy–Khinchin and Lévy–Ito theorems [8–10]. Applied to the output process X this connection can be, *somewhat informally*, described as follows:

'Behind' every output random variable $X(t)$ there exists an inhomogeneous Poisson process $\mathcal{X}(t)$, defined on the real line, which $X(t)$ is its aggregate:

$$X(t) = \sum_{x\in\mathcal{X}(t)} x. \tag{4}$$

The inhomogeneous Poisson process $\mathcal{X}(t)$ is an infinite countable collection of real-valued points, and the aggregate $X(t)$ is convergent if and only if the integral

$$\int_{-\infty}^{\infty} K(t;s)^{\alpha}\,\mathrm{d}s \tag{5}$$

is finite.

The collection $\mathcal{X} = (\mathcal{X}(t))_t$, of inhomogeneous Poisson processes, is the underlying *Poissonian structure* of the output process X. The underlying Poissonian structure $\mathcal{X}$ is unique, and its rigorous construction and description is detailed in [25].

Given a *resolution level* l $(l > 0)$, let $\Pi_l(t)$ denote the number of points of the Poisson process $\mathcal{X}(t)$ which are greater, in absolute value, than the resolution level l. Namely,

$$\Pi_l(t) = \#\{x \in \mathcal{X}(t) \,|\, |x| > l\}. \tag{6}$$

The random process $\Pi_l = (\Pi_l(t))_t$ is the lth *level process* of the underlying Poissonian structure $\mathcal{X}$.

Having defined the level processes, we are now in position to state the following key result, whose proof is given in [25]:

Theorem 1. *The probability generating functions of the multidimensional marginal distributions of the level process* Π_l *are given by*

$$\left\langle z_1^{\Pi_l(t_1)}\cdots z_n^{\Pi_l(t_n)}\right\rangle = \exp\{\Lambda(l)\cdot P_n(t_1,\ldots,t_n;z_1,\ldots,z_n)\} \tag{7}$$

where (i) $\{t_j\}_{j=1}^n$ *are arbitrary time points and* $\{z_j\}_{j=1}^n$ *are the corresponding generating-function variables; (ii)* $\Lambda(l) = al^{-\alpha}$*—the parameters* α *and* a *being, respectively, the input's Lévy exponent and amplitude and (iii)*

$$P_n(t_1, \ldots, t_n; z_1, \ldots, z_n) = \sum_{k=1}^{n} \sum_{1\leqslant j_1<\cdots<j_k\leqslant n} R_k(t_{j_1}, \ldots, t_{j_k}) \cdot (z_{j_1} - 1)\cdots(z_{j_k} - 1) \tag{8}$$

is a polynomial in the variables $\{z_j\}_{j=1}^n$ *whose coefficients are given by*

$$R_k(t_{j_1}, \ldots, t_{j_k}) = \int_{-\infty}^{\infty} \min\{K(t_{j_1}; s), \ldots, K(t_{j_k}; s)\}^{\alpha}\, \mathrm{d}s. \tag{9}$$

We henceforth refer to the sequence of coefficients $\mathcal{R} = \{R_1, R_2, \ldots\}$ as the *Fractal Lévy correlation cascade* of the output process X. The cascade $\mathcal{R}$ is well defined if and only if the integrals appearing in equation (5) are finite for all t. The cascade $\mathcal{R}$ is contingent on Lévy exponent α of the input noise and on the system's integration kernel $K(t; s)$.

Equation (7) is the 'Lévy counterpart' of equation (3). The process distribution of the level processes $\{\Pi_l\}_{l>0}$ is *characterized* by the function $\Lambda(l)$ and by the cascade $\mathcal{R}$.

The exponent on the right-hand side of equation (7) admits an *amplitudal-temporal factorization*—decomposing it into the product of two terms: (i) the function $\Lambda(l)$ which depends on the resolution level l and is independent of the integration kernel $K(t; s)$ and (ii) the polynomial $P_n(t_1, \ldots, t_n; z_1, \ldots, z_n)$ which depends on the first n members $\{R_1, R_2, \ldots, R_n\}$ of the cascade $\mathcal{R}$ and is independent of the resolution level l. The factor $\Lambda(l)$ captures the output's *amplitudal structure*, whereas the polynomial $P_n(t_1, \ldots, t_n; z_1, \ldots, z_n)$ captures the output's *temporal structure*.

The *mean function* and the *auto-covariance function* of the level process Π_l are given, respectively, by $\mu_l(t) = \Lambda(l) \cdot R_1(t)$ and by $C_l(t', t'') = \Lambda(l) \cdot R_2(t', t'')$. Hence, the *auto-correlation function* of the level process Π_l is given by

$$\mathbf{r}(t', t'') = \frac{R_2(t', t'')}{\sqrt{R_1(t')}\sqrt{R_1(t'')}}. \tag{10}$$

The auto-correlation function turns out to be *resolution free*—being independent of the resolution level l.

Since the functions R_1 and R_2 characterize (up to the multiplicative factor $\Lambda(l)$) the mean and the auto-covariance functions of the level processes, and since the first n members $\{R_1, R_2, \ldots, R_n\}$ of the cascade $\mathcal{R}$ characterize (up to the multiplicative factor $\Lambda(l)$) the n-dimensional marginal distributions of the level processes, we conclude that the function R_n can be considered as an '*n-point correlation function*' of the underlying Poissonian structure.

3. Examples

The examples presented below illustrate the theory described above (the detailed calculations are given in [25]). In this section, we make use of the following shorthand notation: (i) $\mathbf{I}\{S\}$:= the indicator function of the set S; (ii) $(\theta)_+ := \max\{0, \theta\}$ (θ real) and (iii) $t_{\min} := \min\{t_1, \ldots, t_n\}$ and $t_{\max} := \max\{t_1, \ldots, t_n\}$.

3.1. Stable motions

Stable motions constitute one of the most elemental classes of stochastic processes. They are self-similar processes [4] with stationary and independent increments [8–10]. In the CLT setting with *infinite variance* stable motions arise as the only possible stochastic-process limits [30]. Stable motions serve as the fundamental model of anomalous diffusion in a broad array of physical systems [11, 12].

Stable motions are the integrals of stable Lévy noises in very same way that Brownian motion is the integral of white noise. In the setting of equation (1) stable motions are the outputs of the integration kernel $K(t; s) = \mathbf{I}\{0 \leqslant s \leqslant t\}$, where time t assumes non-negative values ($t \geqslant 0$). In this case,

$$R_n(t_1, \ldots, t_n) = t_{\min}$$

and

$$\mathbf{r}(t, t + \tau) = \sqrt{\frac{1}{1 + \tau/t}}.$$

The cascade $\mathcal{R}$ turns out to be independent of the Lévy exponent α, and the auto-correlation $\mathbf{r}$ turns out to be identical to the 'standard' auto-correlation of Brownian motion. The latter fact is rather surprising since there is no *a priori* reason why the auto-correlation of a Brownian output (resulting from a white noise input) be identical to the auto-correlation of the underlying Poissonian structure of stable outputs (resulting from stable Lévy noise inputs).

3.2. Stable Ornstein–Uhlenbeck motions

Stable Ornstein–Uhlenbeck (OU) motions are stationary Markov processes whose dynamics are governed by the *Langevin* linear stochastic differential equation $\dot{X} = -\kappa X + \dot{N}$ driven by a stable Lévy noise N (with κ being a positive Hookian parameter) [20–22]. The Langevin equation—a 'cornerstone' stochastic-dynamics equation in Physics—describes the dynamics of a particle trapped in a harmonic potential well and perturbed by an external noise [2].

In the setting of equation (1) stable OU motions are the outputs of the integration kernel $K(t; s) = \mathbf{I}\{s \leqslant t\} \cdot \exp\{s - t\}$, where time t assumes real values ($-\infty < t < \infty$). In this case,

$$R_n(t_1, \ldots, t_n) = \exp\{-\alpha\kappa(t_{\max} - t_{\min})\}$$

and

$$\mathbf{r}(t, t + \tau) = \exp\{-\alpha\kappa\tau\}.$$

The auto-correlation $\mathbf{r}$ of stable OU motions turns out to be identical to the 'standard' auto-correlation of Brownian OU motions. As in the case of stable motions this fact is rather surprising—since there is no *a priori* reason why the auto-correlation of Brownian OU outputs (resulting from a white noise input) be identical to the auto-correlation of the underlying Poissonian structure of stable OU outputs (resulting from stable Lévy noise inputs).

3.3. Stable moving-average motions

Stable moving-average (MA) motions are the generalizations of stable OU motions. They arise in the context of signal processing [16, 31] as outputs of a *convolution filter*, with *impulse-response function* $f(\tau)$ ($\tau \geqslant 0$),[3] 'fed' by the input stable Lévy noise: $X(t) = \int_{-\infty}^{t} f(t - s) N(\mathrm{d}s)$. The resulting stable MA motions are stationary *non-Markov* stochastic processes.

In physics, stable MA motions facilitate the construction of *shot noise* processes exhibiting the *Noah effect* and the *Joseph effect* [32], see [26, 33, 34].

[3] The impulse-response function is non-negative valued and is monotonically decreasing to zero.

In the setting of equation (1) stable MA motions are the outputs of the integration kernel $K(t;s) = \mathbf{I}\{s \leqslant t\} \cdot f(t-s)$, where time t assumes real values ($-\infty < t < \infty$). In this case,

$$R_n(t_1, \ldots, t_n) = F(t_{\max} - t_{\min})$$

and

$$\mathbf{r}(t, t+\tau) = \frac{F(\tau)}{F(0)},$$

where $F(\tau) := \int_\tau^\infty f(u)^\alpha \, \mathrm{d}u$ ($\tau \geqslant 0$).

It is possible to '*reverse engineer*' the auto-correlation $\mathbf{r}$ of stable MA motions: if we wish that $\mathbf{r}(t, t+\tau) = \rho(\tau)$, where $\rho(\tau)$ ($\tau \geqslant 0$) is a smooth function decreasing monotonically from unity to zero, then the impulse-response function should be taken to be $f(\tau) = (-\rho'(\tau))^{1/\alpha}$. We give three 'reverse-engineering' examples:

(i) *Exponential correlations* of the form $\mathbf{r}(t, t+\tau) = \exp\{-b\tau\}$ are obtained by taking exponential impulse-response functions of the form $f(\tau) = \exp\left\{-\frac{b}{\alpha}\tau\right\}$ ($b > 0$).

(ii) *Stretched-exponential correlations* of the form $\mathbf{r}(t, t+\tau) = \exp\{-b\tau^\beta\}$ are obtained by taking impulse-response functions of the form $f(\tau) = \exp\left\{-\frac{b}{\alpha}\tau^\beta\right\}\tau^{(\beta-1)/\alpha}$ ($b > 0$; $0 < \beta < 1$).

(iii) *Power-law correlations* of the form $\mathbf{r}(t, t+\tau) = (1+b\tau)^{-\beta}$ are obtained by taking power-law impulse-response functions of the form $f(\tau) = (1+b\tau)^{-(\beta+1)/\alpha}$ ($b, \beta > 0$).

3.4. Fractional stable motions and noises

The celebrated *fractional Brownian motions*, first introduced by Mandelbrot and Van Ness [35], are self-similar Gaussian processes with stationary and *dependent* increments [4]. In the CLT setting with finite variance and *long-range dependence* [36–38], fractional Brownian motions arise as the only possible stochastic-process limits [30].

Fractional stable motions are the 'Lévy counterparts' of fractional Brownian motions. They are self-similar stable processes with stationary and *dependent* increments [4]. In the CLT setting with *infinite variance* and *long-range dependence*, fractional stable motions arise as the only possible stochastic-process limits [30].

In the setting of equation (1) fractional stable motions are the outputs of the integration kernel $K(t;s) = (t-s)_+^{H-1/\alpha} - (0-s)_+^{H-1/\alpha}$, where time t assumes non-negative values ($t \geqslant 0$). The Lévy exponent α is restricted to the range $1 < \alpha < 2$, and the *Hurst parameter* H—governing the motions' long-range dependence—takes values in the range $1/\alpha < H < 1$. In this case,

$$R_n(t_1, \ldots, t_n) = c \cdot (t_{\min})^{\alpha H},$$

where c is a constant depending on the parameters α and H, and

$$\mathbf{r}(t, t+\tau) = \left(\frac{1}{1+\tau/t}\right)^{\frac{\alpha H}{2}}.$$

The limit $H \to 1/\alpha$ yields the cascade $\mathcal{R}$ (and, consequently, the auto-correlation $\mathbf{r}$) corresponding to stable motions.

Fractional stable noises are the unit-increment sequences of fractional stable motions in very same way that fractional Gaussian noises unit-increment sequences of fractional Brownian motions [4]. In the setting of equation (1) fractional stable noises are the outputs of the integration kernel $K(t;s) = (t-s)_+^{H-1/\alpha} - (t-1-s)_+^{H-1/\alpha}$, where time t assumes integer values ($t = 0, \pm 1, \pm 2, \ldots$). In this case,

$$R_n(t_1, \ldots, t_n) \sim \frac{c_1}{(t_{\max} - t_{\min})^{\alpha(1-H)}}$$

(as $(t_{\max} - t_{\min}) \to \infty$) and

$$\mathbf{r}(t, t+\tau) \sim \frac{c_2}{\tau^{\alpha(1-H)}}$$

(as $\tau \to \infty$), where c_1 and c_2 are constants depending on the parameters α and H.

4. Conclusions

The theory presented considered the integral transform of equation (1) which models a wide range of physical systems convoluting uncorrelated input noises into correlated output processes. There is a profound and marked difference between systems driven by white noise and systems driven by other symmetric stable Lévy noises.

In the former case, the output is Gaussian and its process distribution is characterized by a single function—the output's auto-covariance function. In the latter case, variances are infinite and, consequently, the output's auto-covariance function is undefined.

Nonetheless, in the non-Gaussian case the output process admits an underlying *Poissonian structure* which gives rise to an infinite cascade $\mathcal{R}$ of 'Poissonian correlation functions'. The cascade $\mathcal{R}$ characterizes the output's underlying Poissonian structure and is the 'Lévy counterpart' of the auto-covariance function in the Gaussian case.

The output's correlation structure in the non-Gaussian case is far more rich and complex than in the Gaussian case: captured by a single function in the Gaussian case and by an infinite cascade of functions in the non-Gaussian case. We coin $\mathcal{R}$ the output's *fractal Lévy correlation cascade*.

References

[1] Van Kampen N G 2001 *Stochastic Processes in Physics and Chemistry* (Amsterdam: North-Holland)
[2] Coffey W T, Kalmykov Yu P and Waldron J T 2004 *The Langevin Equation* 2nd edn (Singapore: World Scientific)
[3] Ito K and Hida T 1991 *Gaussian Random Fields* (Singapore: World Scientific)
[4] Embrechts P and Maejima M 2002 *Selfsimilar Processes* (Princeton, NJ: Princeton University Press)
[5] Feller W 1971 *An Introduction to Probability Theory and Its Applications* 2nd edn, vol 2 (New York: Wiley)
[6] Lévy P 1954 *Théorie de L'addition des Variables Aléatoires* (Paris: Gauthier-Villars)
[7] Gnedenko B V and Kolmogorov A N 1954 *Limit Distributions for Sums of Independent Random Variables* (Reading, MA: Addison-Wesley)
[8] Samrodintsky G and Taqqu M S 1994 *Stable Non-Gaussian Random Processes* (London: Chapman and Hall)
[9] Jianicki A and Weron A 1994 *Simulation and Chaotic Behavior of Stable Stochastic Processes* (New York: Dekker)
[10] Applebaum D 2004 *Lévy Processes and Stochastic Calculus* (Cambridge: Cambridge University Press)
[11] Shlesinger M F, Zaslavsky G M and Klafter J 1993 *Nature* **363** 31
[12] Klafter J, Shlesinger M F and Zumofen G 1996 *Phys. Today* **49** 33
[13] Peng C K, Mietus J, Hausdorff J M, Havlin S, Stanley H E and Goldberger A L 1993 *Phys. Rev. Lett.* **70** 1343
[14] Segev R, Benveniste M, Hulata E, Cohen N, Palevski A, Kapon E, Shapira Y and Ben-Jacob E 2002 *Phys. Rev. Lett.* **88** 118102
[15] Sotolongo-Costa O, Antoranz J C, Posadas A, Vidal F and Vazquez A 2000 *Geophys. Rev. Lett.* **27** 1965
[16] Nikias C L and Shao M 1995 *Signal Processing with Alpha-Stable Distributions and Applications* (New York: Wiley)
[17] Mantegna R N and Stanley H E 2000 *An Introduction to Econophysics* (Cambridge: Cambridge University Press)

[18] Bouchaud J P and Potters M 2000 *Theory of Financial Risk* (Cambridge: Cambridge University Press)
[19] Dietlevsen P D 1999 *Phys. Rev.* E **60** 172
[20] Eliazar I and Klafter J 2003 *J. Stat. Phys.* **111** 739
[21] Chechkin A V *et al* 2003 *Phys. Rev.* E **67** 010102(R)
[22] Eliazar I and Klafter J 2005 *J. Stat. Phys.* **119** 165
[23] Chechkin A V, Gonchar V Yu, Klafter J and Metzler R 2005 *Europhys. Lett.* **72** 348
[24] Weron A, Burnecki K, Mercik S and Weron K 2005 *Phys. Rev.* E **71** 016113
[25] Eliazar I and Klafter J 2007 Correlation cascades of Lévy-driven random processes *Physica* A **376** 1
[26] Eliazar I and Klafter J 2007 On the analysis of shot noise displaying simultaneously the Noah and Joseph effects *Phys. Rev.* E **75** 031108
[27] Cambanis S, Podgórski K and Weron A 1995 *Stud. Math.* **115** 109
[28] Rosinski J and Zak T 1997 *J. Theor. Prob.* **10** 73
[29] Kingman J F C 1993 *Poisson Processes* (Oxford: Oxford University Press)
[30] Whitt W 2002 *Stochastic-Process Limits* (Berlin: Springer)
[31] Brémaud P 2002 *Mathematical Principles of Signal Processing* (Berlin: Springer)
[32] Mandelbrot B B and Wallis J R 1968 *Water Resour. Res.* **4** 909
[33] Lowen S B and Teich M C 1989 *Phys. Rev. Lett.* **63** 1755
[34] Lowen S B and Teich M C 1990 *IEEE Trans. Inform. Theory* **36** 1302
[35] Mandelbrot B B and Van Ness J W 1968 *SIAM Rev.* **10** 422
[36] Cox D R 1984 Long-range dependence: a review *Statistics: An Appraisal* ed H A David and H T David (Iowa City, IO: University of Iowa Press) pp 55–74
[37] Taqqu M and Oppenheim G (ed) 2002 *Theory and Applications of Long-range Dependence* (Basle: Birkhauser)
[38] Rangarajan G and Ding M (ed) 2003 *Processes with Long-range Correlations: Theory and Applications (Lecture Notes in Physics* vol 621*)* (Berlin: Springer)